A Minimal FDTD (Yee) Method for Simulating Light

Maxwell updates, stability, and boundaries

February 22, 2026

Abstract

Finite-difference time-domain (FDTD) methods simulate electromagnetic waves by discretizing Maxwell's equations on a staggered grid (the Yee lattice) and marching the electric and magnetic fields forward in time. This note summarizes a standard 2D FDTD scheme (TM_z) with explicit update equations, the Courant–Friedrichs–Lewy (CFL) stability condition, a simple source injection, and boundary treatment via absorbing layers (PML overview).

Contents

1 Maxwell equations (time domain)

In a linear, isotropic medium with conductivity σ (optionally 0), permittivity ϵ, permeability μ:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t},$$
$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E}.$$

FDTD uses a leapfrog time stepping: **E** and **H** are stored at interleaved times.

2 Yee grid idea (staggering)

The Yee scheme staggers field components in space so that discrete curls are centered. In 2D TM_z polarization we evolve

$$E_z(x,y,t), \qquad H_x(x,y,t), \qquad H_y(x,y,t),$$

assuming $\partial/\partial z = 0$ and $E_x = E_y = H_z = 0$.

3 2D TM_z FDTD update equations

3.1 Continuous TM_z form

For TM_z,

$$\begin{aligned}
\frac{\partial H_x}{\partial t} &= -\frac{1}{\mu}\frac{\partial E_z}{\partial y}, \\
\frac{\partial H_y}{\partial t} &= \frac{1}{\mu}\frac{\partial E_z}{\partial x}, \\
\frac{\partial E_z}{\partial t} &= \frac{1}{\epsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) - \frac{\sigma}{\epsilon}E_z.
\end{aligned}$$

3.2 Discrete indices and steps

Let spatial steps be $\Delta x, \Delta y$ and time step Δt. Use integer indices (i, j) for grid cells. The canonical Yee staggering is

$$E_z^n(i,j), \quad H_x^{n+\frac{1}{2}}(i, j+\tfrac{1}{2}), \quad H_y^{n+\frac{1}{2}}(i+\tfrac{1}{2}, j).$$

3.3 Update: magnetic fields (half step)

$$\begin{aligned} H_x^{n+\frac{1}{2}}\big(i, j+\tfrac{1}{2}\big) &= H_x^{n-\frac{1}{2}}\big(i, j+\tfrac{1}{2}\big) - \frac{\Delta t}{\mu\,\Delta y}\Big(E_z^n(i,j+1) - E_z^n(i,j)\Big), \\ H_y^{n+\frac{1}{2}}\big(i+\tfrac{1}{2}, j\big) &= H_y^{n-\frac{1}{2}}\big(i+\tfrac{1}{2}, j\big) + \frac{\Delta t}{\mu\,\Delta x}\Big(E_z^n(i+1,j) - E_z^n(i,j)\Big). \end{aligned}$$

3.4 Update: electric field (full step)

Define the discrete curl term at (i, j):

$$(\boldsymbol{\nabla} \times \mathbf{H})_z \approx \frac{H_y^{n+\frac{1}{2}}(i+\frac{1}{2}, j) - H_y^{n+\frac{1}{2}}(i-\frac{1}{2}, j)}{\Delta x} - \frac{H_x^{n+\frac{1}{2}}(i, j+\frac{1}{2}) - H_x^{n+\frac{1}{2}}(i, j-\frac{1}{2})}{\Delta y}.$$

With conductivity, a common explicit update is

$$E_z^{n+1}(i,j) = c_a(i,j)\, E_z^n(i,j) + c_b(i,j)\, (\boldsymbol{\nabla} \times \mathbf{H})_z,$$

where

$$c_a = \frac{1 - \frac{\sigma \Delta t}{2\epsilon}}{1 + \frac{\sigma \Delta t}{2\epsilon}}, \qquad c_b = \frac{\Delta t/\epsilon}{1 + \frac{\sigma \Delta t}{2\epsilon}}.$$

For $\sigma = 0$, this reduces to the familiar centered update $E^{n+1} = E^n + (\Delta t/\epsilon)(\boldsymbol{\nabla} \times H)_z$.

4 Stability: CFL condition

For a homogeneous medium with wavespeed $c = 1/\sqrt{\mu\epsilon}$, the 2D CFL condition is

$$\Delta t \leq \frac{1}{c} \frac{1}{\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}}}.$$

In 3D it becomes

$$\Delta t \leq \frac{1}{c} \frac{1}{\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}}}.$$

5 Sources (one simple approach)

5.1 Hard source (overwrite)

Pick a grid point (i_s, j_s) and set

$$E_z^n(i_s, j_s) \leftarrow s(t_n), \qquad t_n = n\Delta t,$$

for a chosen waveform (Gaussian pulse, sinusoid, Ricker wavelet, etc.).

5.2 Soft source (add)

Alternatively add into the update:

$$E_z^{n+1}(i_s, j_s) \leftarrow E_z^{n+1}(i_s, j_s) + s(t_{n+1}).$$

6 Boundaries and absorbing layers

6.1 Why you need absorption

Finite computational windows reflect waves unless boundary conditions are designed to absorb outgoing radiation.

6.2 PML (perfectly matched layer) overview

A PML introduces an artificial absorbing region (often with graded conductivities) designed to minimize reflections. In practice, one typically:

- Adds layers of thickness N_{PML} cells around the domain;
- Uses spatially varying $\sigma_x(x), \sigma_y(y)$ (and sometimes κ and α parameters);
- Splits fields (Berenger PML) or uses uniaxial PML (UPML) / convolutional PML (CPML).

The detailed CPML update is longer; the key is that it modifies the curl terms with auxiliary "memory" variables so that outgoing waves decay exponentially within the layer.

7 A minimal algorithm (pseudo-steps)

1. Initialize $E_z^0, H_x^{-1/2}, H_y^{-1/2}$ (often all zeros).

2. For each time step $n = 0, 1, 2, \ldots$:
   (a) Update $H_x^{n+1/2}, H_y^{n+1/2}$ from E_z^n.
   (b) Update E_z^{n+1} from $H^{n+1/2}$ (plus optional conductivity terms).
   (c) Inject sources into E_z (hard or soft).
   (d) Apply boundary/PML updates.

8 Extensions (what to add next)

- Full 3D update equations $(E_x, E_y, E_z, H_x, H_y, H_z)$.
- Dispersive media (Drude/Lorentz ADE or convolution).
- Anisotropic ϵ (useful for birefringent materials, including liquid crystals).
- Total-field/scattered-field (TFSF) plane-wave injection.
- Near-to-far-field transforms and scattering parameters.