Group Theory and Homotopy Groups

A quick, equation-heavy reference for physics

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Abstract

This is a standalone collection of group theory and homotopy material (Lie groups and Lie algebras, representations, exact sequences, and standard computations of π_n) with a focus on the formulas most often used in gauge theory, defects, and topological terms in QFT.

Contents

1 Groups, actions, and representations

1.1 Groups and homomorphisms

A group $(G, \cdot)$ has identity e and inverses. A homomorphism $\varphi : G \to H$ satisfies

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2).$$

Kernel and image:

$$\ker(\varphi) = \{g \in G : \varphi(g) = e\}, \qquad \mathrm{im}(\varphi) = \varphi(G).$$

1.2 Group actions

A (left) action of G on X is $G \times X \to X$, $(g, x) \mapsto g \cdot x$, with

$$e \cdot x = x, \qquad (g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x).$$

Orbits and stabilizers:

$$G \cdot x = \{g \cdot x\}, \qquad G_x = \{g \in G : g \cdot x = x\}.$$

1.3 Representations

A representation is a homomorphism $\rho : G \to GL(V)$. For Lie groups, differentiating at the identity gives a Lie algebra representation

$$\rho_* : \mathfrak{g} \to \mathfrak{gl}(V), \qquad \rho_*([X,Y]) = [\rho_*(X), \rho_*(Y)].$$

2 Lie groups and Lie algebras

2.1 Lie algebra from matrices

For a matrix Lie group $G \subseteq GL(n,\mathbb{C})$, the Lie algebra is

$$\mathfrak{g} = \{X \in M_n(\mathbb{C}) : e^{tX} \in G \ \forall t\}.$$

The bracket is the commutator $[X,Y] = XY - YX$.

2.2 Adjoint and coadjoint actions

$${}_g(X) = gXg^{-1}, \qquad {}_X(Y) = [X,Y].$$

The coadjoint action on $\mathfrak{g}^*$ is $({}^*_g\lambda)(X) = \lambda({}_{g^{-1}}X)$.

2.3 Structure constants and normalization

Choose a basis T^a:

$$[T^a, T^b] = f^{ab}{}_c T^c.$$

In a representation one often sets

$$\mathrm{Tr}\Big(T^a T^b\Big) = \kappa\, \delta^{ab}, \qquad (T^a)^\dagger = T^a \text{ (compact groups)}.$$

3 Exact sequences and covering spaces

3.1 Short exact sequences

A short exact sequence

$$1 \to A \xrightarrow{i} B \xrightarrow{p} C \to 1$$

means i is injective, p is surjective, and $\mathrm{im}(i) = \ker(p)$. This packages extensions.

3.2 Universal covers (example)

$SU(2)$ is the universal cover of $SO(3)$:

$$1 \to \mathbb{Z}_2 \to SU(2) \to SO(3) \to 1,$$

hence

$$\pi_1(SO(3)) \cong \mathbb{Z}_2, \qquad \pi_1(SU(2)) = 0.$$

4 Homotopy groups: definitions

4.1 Based homotopy groups

For a pointed space (X, x_0),

$$\pi_n(X, x_0) := \{f : (\mathbb{S}^n, *) \to (X, x_0)\}/\sim$$

where $\sim$ is based homotopy. For $n \geq 1$ these are groups, and for $n \geq 2$ they are abelian.

4.2 Relative homotopy and pairs

For a pair (X, A) with basepoint in A, the relative groups $\pi_n(X, A)$ fit into a long exact sequence (LES).

5 Fibrations and the long exact sequence

Given a fibration $F \hookrightarrow E \twoheadrightarrow B$, there is an LES:

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$$

This is among the most practical tools for computing π_n.

5.1 Example: $U(1) \hookrightarrow SU(2) \twoheadrightarrow S^2$

The quotient $SU(2)/U(1) \cong S^2$ yields a fibration

$$U(1) \hookrightarrow SU(2) \twoheadrightarrow S^2.$$

Using $\pi_2(SU(2)) = 0$ and $\pi_1(SU(2)) = 0$, the LES gives

$$\pi_2(S^2) \cong \pi_1(U(1)) \cong \mathbb{Z}.$$

6 Standard computations and facts

6.1 Spheres

$$\pi_1(S^1) \cong \mathbb{Z}, \qquad \pi_n(S^n) \cong \mathbb{Z} \ (n \geq 1), \qquad \pi_1(S^n) = 0 \ (n \geq 2).$$

6.2 Lie groups (selected low homotopy)

Common values (for $N \geq 2$):

$$\pi_1(SU(N)) = 0, \qquad \pi_3(SU(N)) \cong \mathbb{Z}.$$

For $N \geq 3$:

$$\pi_1(SO(N)) \cong \mathbb{Z}_2.$$

6.3 Homogeneous spaces and defects

If G breaks to H, the vacuum manifold is $\mathcal{M} \cong G/H$. Defects are classified by

$$\pi_k(\mathcal{M})$$

depending on codimension.

7 A minimal "physics bridge" appendix

7.1 Gauge fields and characteristic classes (one page)

Write a connection A and curvature $F = dA + A \wedge A$. A basic topological density in 4D is

$$\mathrm{Tr}(F \wedge F),$$

with instanton number (normalization depending on conventions)

$$k = \frac{1}{8\pi^2} \int_{M_4} \mathrm{Tr}(F \wedge F) \in \mathbb{Z},$$

and corresponding θ-term

$$S_\theta = i\theta\, k.$$

7.2 A quick picture: maps at infinity

For finite-action configurations in 4D Euclidean Yang–Mills, at $|x| \to \infty$ one effectively gets a map

$$S^3_\infty \to G,$$

classified by $\pi_3(G)$, e.g. $\pi_3(SU(N)) \cong \mathbb{Z}$.