Disclinations and Topological Defects in Liquid Crystals

Importance, geometry, and homotopy classification

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Abstract

Liquid crystals support robust topological defects whose classification is controlled by the topology of the order-parameter space. This note focuses on disclinations (line defects) and related textures in nematic and smectic phases, highlighting why they matter (optics, rheology, response to fields, active matter, defect-mediated transitions) and presenting the standard mathematical tools: order parameters, homotopy groups, winding/charge, and energetic models (Frank–Oseen and Landau–de Gennes).

Contents

1 Why defects in liquid crystals matter (physics motivation)

- **Optical and electro-optical response:** director distortions modify birefringence and light propagation.
- **Mechanical response:** defects store elastic energy and control plasticity-like flow under shear.
- **Field-driven manipulation:** electric/magnetic fields move and create/annihilate defects.
- **Active nematics:** defects become dynamical quasi-particles (creation/annihilation, chaotic flows).
- **Topology:** defect charges constrain which configurations can relax without singular events.

2 Order parameters

2.1 Nematic director and head-tail symmetry

A nematic is described (in the simplest model) by a *director* field

$$\mathbf{n}(\mathbf{x}) \in \mathbb{S}^2, \qquad \mathbf{n} \sim -\mathbf{n}.$$

Thus the order-parameter space is

$$\mathcal{M}_{\text{nem}} \cong \mathbb{S}^2/(\mathbf{n} \sim -\mathbf{n}) \cong \mathbb{RP}^2.$$

2.2 Q-tensor order parameter (Landau–de Gennes)

In continuum theory one often uses a traceless symmetric tensor

$$Q_{ij} = S\left(n_i n_j - \frac{1}{3}\delta_{ij}\right), \qquad \operatorname{Tr} Q = 0,$$

with scalar order parameter S.

3 Disclinations as line defects

3.1 Geometric definition

A *disclination* is a line defect where the director is ill-defined along a curve, typically because the director rotates by a nontrivial amount around any small loop linking the line.

3.2 2D picture: winding number for an angle field

In 2D (or for planar textures) write $\mathbf{n} = (\cos\theta, \sin\theta)$. Because $\mathbf{n} \sim -\mathbf{n}$, the angle is identified as

$$\theta \sim \theta + \pi,$$

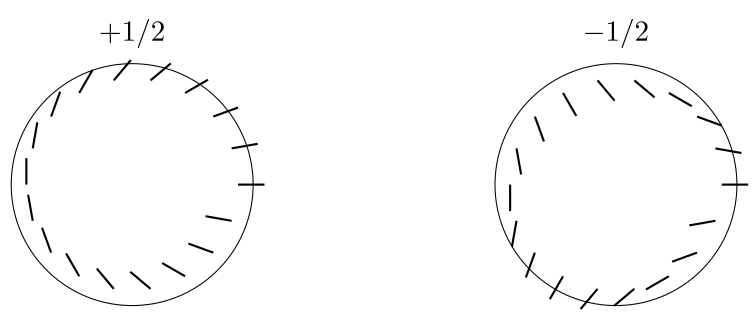
so half-integer windings are allowed. Define the charge (strength)

$$s := \frac{1}{2\pi} \oint_\gamma d\theta,$$

for a loop γ around the defect. Then

$$s \in \frac{1}{2}\mathbb{Z} \qquad \text{(planar nematic disclinations).}$$

3.3 A small figure: $+1/2$ and $-1/2$ disclinations



4 Homotopy classification

4.1 General principle

If the medium has vacuum manifold $\mathcal{M}$, then defects of codimension $k+1$ are classified (in the simplest setting) by

$$\pi_k(\mathcal{M}).$$

For line defects in 3D, $k = 1$ and one uses $\pi_1(\mathcal{M})$.

4.2 Fundamental group of $\mathbb{RP}^2$ and nematic line defects

For nematics $\mathcal{M} \cong \mathbb{RP}^2$, and

$$\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2.$$

So *topologically stable* nematic line defects carry a $\mathbb{Z}_2$ charge: two identical $\mathbb{Z}_2$ defects can annihilate.

4.3 Point defects (hedgehogs)

Point defects in 3D are classified by $\pi_2(\mathcal{M})$. For $\mathbb{RP}^2$,

$$\pi_2(\mathbb{RP}^2) \cong \mathbb{Z},$$

so hedgehogs carry an integer charge (degree) in idealized settings. In practice, core structure and biaxial escape can modify how this is realized in a given material model.

5 Energetics: Frank–Oseen theory

For a unit director field $\mathbf{n}(\mathbf{x})$ with $\mathbf{n} \sim -\mathbf{n}$, the Frank free energy is

$$F[\mathbf{n}] = \frac{1}{2} \int d^3x \left(K_1(\nabla \cdot \mathbf{n})^2 + K_2(\mathbf{n} \cdot \nabla \times \mathbf{n})^2 + K_3 |\mathbf{n} \times (\nabla \times \mathbf{n})|^2 \right),$$

with splay, twist, and bend moduli K_1, K_2, K_3. In the one-constant approximation $K_1 = K_2 = K_3 = K$,

$$F = \frac{K}{2} \int d^3x \, (\partial_i n_j)(\partial_i n_j).$$

5.1 Energy scaling near a 2D disclination

For a planar texture with $\theta = s\varphi$ (polar angle φ), one has $|\nabla\theta| \sim |s|/r$ so

$$F \sim \frac{K}{2} \int_a^R r \, dr \int_0^{2\pi} d\varphi \, \frac{s^2}{r^2} = \pi K s^2 \ln\left(\frac{R}{a}\right),$$

with core cutoff a and system size R. This logarithmic divergence is why defects interact and why confinement/annihilation is energetically favorable in many settings.

6 Landau–de Gennes and defect cores

In Q-tensor theory, one writes a bulk+gradient functional such as

$$F[Q] = \int d^3x \left(\frac{A}{2} \operatorname{Tr}(Q^2) + \frac{B}{3} \operatorname{Tr}(Q^3) + \frac{C}{4} (\operatorname{Tr}(Q^2))^2 + \frac{L}{2} (\partial_k Q_{ij})(\partial_k Q_{ij}) \right),$$

which allows the magnitude S (and possible biaxiality) to vary, regularizing defect cores.

7 Smectics (brief remarks)

Smectics have a layered structure often described by a phase field ϕ with layers $\phi = \text{const}$. Dislocations and disclinations intertwine with the topology of the foliation; the relevant "order parameter space" involves both a phase and a layer normal, and the classification can be subtler than the nematic $\mathbb{RP}^2$ case.

8 Summary

- Nematic order has $\mathcal{M} \cong \mathbb{RP}^2$, giving $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$ line defects and $\pi_2(\mathbb{RP}^2) \cong \mathbb{Z}$ point defects.
- Planar disclinations allow half-integer strengths $s \in \frac{1}{2}\mathbb{Z}$ due to $\mathbf{n} \sim -\mathbf{n}$.
- Frank–Oseen theory gives logarithmic energy scaling for isolated 2D disclinations, explaining defect interactions.
- Q-tensor Landau–de Gennes theory resolves defect cores by allowing S (and biaxiality) to vary.