Optical (Photonic) Integrated Circuits: A Mathematical Primer

Waveguides, couplers, resonators, and system models

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Abstract

Photonic integrated circuits (PICs) implement optical functions (routing, filtering, modulation, detection) on-chip using waveguides, couplers, and resonators in material platforms such as silicon photonics, SiN, III–V semiconductors, and thin-film $LiNbO_3$. This note emphasizes the core equations used to model PIC building blocks: guided modes from Maxwell's equations, coupled-mode theory, scattering/transfer matrices, ring resonators, Mach–Zehnder interferometers, and basic link metrics (insertion loss, extinction ratio, Q factor, bandwidth).

Contents

1 Maxwell and guided modes

In source-free linear media,

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \qquad \nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}.$$

Assume harmonic time dependence $e^{-i\omega t}$. Then

$$\nabla \times \nabla \times \mathbf{E} = k_0^2 \, n^2(\mathbf{r}) \, \mathbf{E}, \qquad k_0 = \frac{\omega}{c}.$$

1.1 Waveguide eigenmodes and effective index

For a straight waveguide invariant along z, guided modes take the form

$$\mathbf{E}(\mathbf{r}) = \mathbf{e}(x, y) \, e^{i\beta z}, \qquad n_{\text{eff}} := \frac{\beta}{k_0}.$$

Propagation can be expressed as

$$a(z) = a(0) e^{-(\alpha/2)z} e^{i\beta z},$$

where α is the power attenuation coefficient (Np/m or dB/m).

2 Phase, group index, and dispersion

The group velocity and group index are

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}\beta}, \qquad n_g = \frac{c}{v_g} = c\,\frac{\mathrm{d}\beta}{\mathrm{d}\omega}.$$

The dispersion parameter often used in integrated optics is related to $\frac{\mathrm{d}^2\beta}{\mathrm{d}\omega^2}$ (or equivalently $\frac{\mathrm{d}^2 n_{\text{eff}}}{\mathrm{d}\lambda^2}$).

3 Scattering matrices (PIC viewpoint)

Linear, time-invariant, passive reciprocal components are often modeled by a (frequency-dependent) scattering matrix $S(\omega)$:

$$\mathbf{b} = S(\omega)\,\mathbf{a},$$

where $\mathbf{a}$ and $\mathbf{b}$ are vectors of incoming/outgoing complex wave amplitudes.

3.1 Directional coupler as a 2×2 unitary

An ideal lossless symmetric coupler is

$$S_{\text{dc}} = \begin{pmatrix} t & i\kappa \\ i\kappa & t \end{pmatrix}, \qquad |t|^2 + |\kappa|^2 = 1.$$

The power coupling ratio is $|\kappa|^2$.

4 Coupled-mode theory (CMT)

4.1 Two coupled waveguides

For slowly-varying amplitudes $a_1(z), a_2(z)$ (identical guides, phase-matched),

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = i\begin{pmatrix} 0 & \kappa \\ \kappa & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Solution:

$$a_1(z) = a_1(0)\cos(\kappa z) + i a_2(0)\sin(\kappa z), \qquad a_2(z) = i a_1(0)\sin(\kappa z) + a_2(0)\cos(\kappa z).$$

Full power transfer occurs at coupling length $L_\pi = \pi/(2\kappa)$.

5 Mach–Zehnder interferometer (MZI)

An MZI consists of two couplers separated by two arms with phase difference $\Delta\phi$. In the ideal balanced case (50/50 couplers), the normalized output intensity can be written as

$$I_{\text{out}} \propto \frac{1}{2}\left(1+\cos\Delta\phi\right) = \cos^2\left(\frac{\Delta\phi}{2}\right),$$

with

$$\Delta\phi = \frac{2\pi}{\lambda}\,\Delta n_{\text{eff}}\,L.$$

Electro-optic or thermo-optic tuning changes n_{eff} and hence $\Delta\phi$.

6 Ring resonators

6.1 Resonance condition

For a ring of round-trip length L and propagation constant $\beta(\omega)$,

$$\beta(\omega_m)L = 2\pi m, \qquad m \in .$$

The free spectral range (FSR) in frequency is approximately

$$\text{FSR} \approx \frac{v_g}{L} = \frac{c}{n_g L}.$$

6.2 All-pass ring: a standard transfer function

With amplitude self-coupling t and round-trip amplitude factor $a = e^{-(\alpha L)/2}$, a common normalized field transfer function is

$$H(\omega) = \frac{t - ae^{-i\phi(\omega)}}{1 - tae^{-i\phi(\omega)}}, \qquad \phi(\omega) = \beta(\omega)L.$$

The intensity response is $|H(\omega)|^2$ and shows narrow notches near resonance depending on coupling and loss.

6.3 Q factor and linewidth

For a resonance at ω_0 with full-width at half-maximum (FWHM) $\Delta\omega$,

$$Q = \frac{\omega_0}{\Delta\omega}.$$

7 Modulators and detectors (high-level models)

7.1 Phase modulator

For a phase shifter of length L,

$$\Delta\phi(V) = \pi \frac{V}{V_\pi}, \qquad V_\pi L \text{ is a common figure of merit.}$$

7.2 Photodiode

A simple responsivity model:

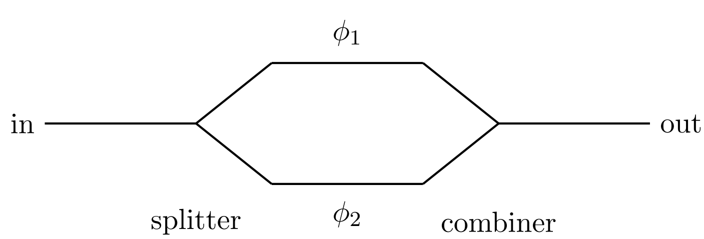
$$I_{\mathrm{pd}} = \mathcal{R}\, P_{\mathrm{opt}},$$

where $\mathcal{R}$ is responsivity (A/W).

8 Loss, crosstalk, and basic metrics

- **Insertion loss (IL):** ratio of output to input power (in dB).
- **Extinction ratio (ER):** $10 \log_{10}(P_{\mathrm{on}}/P_{\mathrm{off}})$.
- **Propagation loss:** modeled by $P(z) = P(0)e^{-\alpha z}$.
- **Bend loss:** grows as bend radius decreases (platform-dependent).

9 A small “PIC block diagram” figure



This MZI-like layout illustrates how many PICs are assembled from interference plus tunable phases.

10 Where to go next

- Multi-mode interference (MMI) couplers and self-imaging.
- Arrayed waveguide gratings (AWGs) and phased arrays.

- Full-vector mode solvers and effective index method.
- Time-domain simulation: FDTD/FEM for layout-level verification.
- Nonlinear optics on chip: Kerr ($\chi^{(3)}$), $\chi^{(2)}$ in $LiNbO_3$, parametric processes.